Some comments about practice exam.

in Problem 4: the notion of mesh(P) for a partition

$P = \{t_k, 0 \le k \le n\}$ is defined by

$$\text{mesh}(P) = \max\{t_k - t_{k-1}, \ k = 1, 2, \dots, n\}$$

example: if $P = \{k/n, \ k = 0, 1, \dots, n\}$

$$\Rightarrow \quad t_k - t_{k-1} = \frac{k}{n} - \frac{(k-1)}{n} = \frac{1}{n}$$

$$\Rightarrow \quad \text{mesh}(P) = \frac{1}{n}.$$

Problem 5: helpful to look at Problem 33.14

That problem proves a generalization of Mean Value Theorem for integrals:

let $g(t) \geq 0$ for $a \leq t \leq b$,

f, g continuous function

$\Rightarrow$ there exists $x \in (a,b)$ such that

$$\int_a^b f(t) g(t)\, dt = f(x) \int_a^b g(t)\, dt$$

idea for solution of Problem 33.14:

f cont. $\Rightarrow$ reaches its max and min in $[a,b]$

$\Rightarrow \exists\, x_0$ s.t. $f_{min} = f(x_0) \leq f(x) \quad \forall\, x \in [a,b]$

$\exists\, y_0$ s.t. $f_{max} = f(y_0) \geq f(x) \quad \forall\, x \in [a,b]$

$g(t) \geq 0$

$\Rightarrow \quad f_{min}\, g(t) \leq f(t)\, g(t) \leq f_{max}\, g(t)$ for all $t \in [a,b]$

$\Rightarrow \quad \int \underbrace{f_{min}}_{f(x_0)} g(t)\,dt \leq \int f(t) g(t)\,dt \leq \int \underbrace{f_{max}}_{f(y_0)} g(t)\,dt$

$f(x_0) \int g(t)\,dt \leq \int f(t) g(t)\,dt \leq f(y_0) \int g(t)\,dt$

$\Rightarrow \quad f(x_0) \leq \dfrac{\int f(t) g(t)\,dt}{\int g(t)\,dt} \leq f(y_0)$

[$\int g(t)\,dt \geq 0$

if $\int g(t)\,dt = 0 \Rightarrow$ claim trivially true as then $g(t) = 0$ for all $t \in [a,b]$ ($g(t)$ is continuous)

can assume $\int g(t)\,dt > 0$

By mean value theorem for continuous functions

$\Rightarrow$ $\exists$ x between x_0 and y_0 such that

$$f(x) = \frac{\int_a^b f(t)\, g(t)\, dt}{\int_a^b g(t)\, dt}$$

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possible integration problem:

show integrability or non integrability using Darboux sums:

integrable: $\quad L(f) = \sup\limits_{\substack{P \text{ any partition} \\ \text{of } [a,b]}} L(f,P) = U(f) = \inf\limits_{P \text{ partition.}} (U,P)$

$\Longleftrightarrow$ for every $\varepsilon > 0$ $\exists$ a partition P s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

not integrable $\Leftrightarrow$ $\exists\, \varepsilon > 0$ s.t.

$$U(f,P) - L(f,P) > \varepsilon$$

for any partition P.

Fundamental Theorem of Calculus I & II

FTC I: If g differentiable such that g' is integrable

$$\Rightarrow \int_a^b g'(x)\,dx = g(b) - g(a)$$

FTC II: If f integrable on $[a,b]$. and $F(x) = \int_a^x f(t)\,dt$

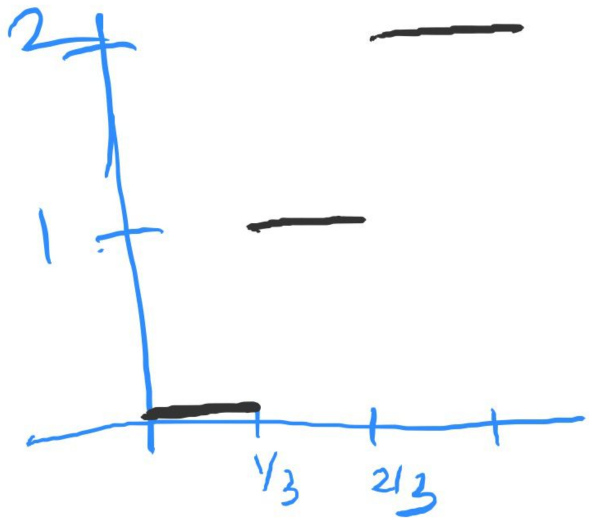
$\Rightarrow$ F is continuous

If f is continuous at $x_0 \in (a,b)$

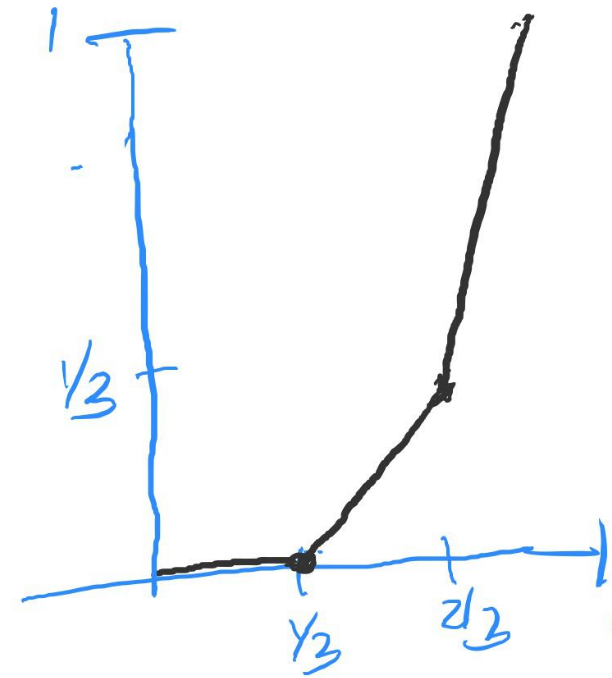
$\Rightarrow$ F differentiable at x_0 and $F'(x_0) = f(x_0)$

Example:

$$f(x) = \begin{cases} 0 & 0 \le x < 1/3 \\ 1 & 1/3 \le x < 2/3 \\ 2 & 2/3 \le x \le 1 \end{cases}$$



$y = f(x)$



check:
$$F(x) = \begin{cases} 0 & 0 \le x \le 1/3 \\ x - \frac{1}{3} & 1/3 < x \le 2/3 \\ 2(x - \frac{1}{3}) + \frac{1}{3} & 2/3 < x \le 1 \end{cases}$$

$\Rightarrow$ F differentiable for all $x \neq \underbrace{1/3 \text{ or } 2/3}_{\text{points where } f \text{ not continuous}}$

Taylor's Theorem:

Determine for which x a function is given by its Taylor series.

$\Longleftrightarrow$ determine for which x the remainder $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k \to 0$ if $n \to \infty$

Taylor's Theorem

$\exists\, y$ between c and x such that

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$$

Ex. $f(x) = e^x \Longrightarrow f^{(n)}(x) = e^x$

fix x. (say $x > 0$)

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n$$

If $0 < y < x \Longrightarrow f^{(n)}(y) = e^y \leq e^x$

$$\Longrightarrow |R_n(x)| \leq \frac{e^x}{n!} x^n$$

have shown before:

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{for any } x$$

$\Longrightarrow R_n(x) \to 0$ for $n \to \infty$

Result: $$e^x = \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!} x^k}_{\text{Taylor series for } e^x.} \quad \text{for any } x$$

other possible problems:

- uniform convergence vs pointwise convergence
- behavior of integrals under uniform/pointwise convergence
  - continuous functions
  - differentiable "
  - integrals
- mean value theorems

if $f: [a,b] \to \mathbb{R}$ cont, differentiable in (a,b) $\Big\}$ $\exists\, x \in (a,b)$ s.t. $f'(x) = \dfrac{f(b)-f(a)}{b-a}$